Two-dimensional von Neumann–Wigner potentials with a multiple positive eigenvalue

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Let H be a Schrödinger operator

$$H = -\Delta + U$$

with a potential $U(x)$, on $\mathbb{R}^N$, decaying at infinity. The potential U is called a *von Neumann–Wigner* potential if H has a positive eigenvalue with an eigenfunction from $L_2(\mathbb{R}^N)$, i.e. there is a point of discrete spectrum which is embedded into the absolutely continuous spectrum.

The very first example of such a potential was constructed by von Neumann and Wigner [1]. They found a three-dimensional rotation-symmetric nonsingular potential $U(r)$ with the following asymptotic behavior (a computational mistake in [1] reported later is corrected here):

$$U(r) = -\frac{8\sin 2r}{r} + O(r^{-2}) \quad \text{as } r = |x| \to \infty,\ x \in \mathbb{R}^3.$$

The Schrödinger operators with $U(x) = o(1/|x|)$ as $x \to \infty$ have no positive eigenvalues [2].

In the present note we construct multiparametric families of explicit two-dimensional potentials which decay as $1/|x|$ and have a multiple positive eigenvalue. To our knowledge these are the first examples of such potentials.

We use the method introduced in [3, 4] where two-dimensional Schrödinger operators with fast decaying potentials and multidimensional kernels were constructed. This method is based on the Moutard transformation, of two-dimensional Schrödinger operators, which is as follows: let ω be a formal solution of the equation

$$H\omega = \big(-\Delta + U(x,y) \big)\omega = 0, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1}$$

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The Moutard transformation corresponding to H and ω gives a new Schrödinger operator

$$\widetilde{H} = -\Delta + \widetilde{U}, \quad \widetilde{U} = U - 2\Delta \log \omega$$

such that if φ satisfies $H\varphi = 0$, then a function θ determined modulo $\frac{\text{const}}{\omega}$ by the consistent system

$$(\omega\theta)_x = -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \qquad (\omega\theta)_y = \omega^2 \left(\frac{\varphi}{\omega}\right)_x, \tag{2}$$

satisfies

$$\widetilde{H}\theta = 0.$$

So there are maps

$$U \to M_\omega(U) = U - 2\Delta \log \omega, \quad \varphi \to S_\omega(\varphi) = \{\theta + \frac{C}{\omega}, C \in \mathbb{C}\}.$$

Let us consider an operator H and a pair of solutions to (1): ω_1 and ω_2. For every $\theta_1 \in S_{\omega_1}(\omega_2)$ there is a function (a result of double iteration of the Moutard transformation)

$$\widehat{U} = M_{\theta_1} M_{\omega_1}(U) - U = -2\Delta \log(\theta_1 \omega_1). \tag{3}$$

The result of this iteration depends on the choice of $\theta_1 \in S_{\omega_1}(\omega_2)$, i.e., on the integration constant C in (2). Moreover the functions

$$\psi_1 = \frac{1}{\theta_1}, \quad \psi_2 = \frac{\omega_2}{\omega_1 \theta_1}$$

satisfy the equation

$$(-\Delta + M_{\theta_1} M_{\omega_1}(U))\psi = 0.$$

In contrast to [3, 4] where such a double iteration was applied to the case $U = 0$, we apply it to the constant potential $U = -k^2, k \in \mathbb{R}$. Therewith ω_1 and ω_2 have to satisfy to the Helmholtz equation

$$-\Delta\omega = k^2\omega.$$

A large set of solutions to this equation is given by functions of the form

$$\text{Re}\left[\frac{\partial^m}{\partial\lambda^m} e^{i\frac{k}{2}(\lambda z + \frac{\bar{z}}{\lambda})}\right], \quad \lambda \in \mathbb{C}, \quad m = 0, 1, 2, \dots, \tag{4}$$

and their linear combinations.

For simplicity we consider the case $k^2 = 1$ and demonstrate the method by one explicit example.

Theorem 1 *Let $U = -1$ and*

$$\omega_1 = x^2 \cos y - y \sin y + y^2 \sin x + x \cos x, \quad \omega_2 = 4(y \cos x + x \sin y), \quad x, y \in \mathbb{R}.$$

Then the two-dimensional potential $\widehat{U}$ *takes the form*

$$\widehat{U} = \frac{P}{Q^2}$$

where

$$Q = \omega_1\theta_1 = -x^4 - y^4 - 4x^2 y \sin x \sin y + x^2\big(-8\cos y \sin x - 2\sin^2 y - 1\big) +$$
$$+4xy^2 \cos x \cos y - 16xy \cos x \sin y + 2x\cos x\big(-8\cos y - \sin x\big) +$$
$$+y^2\big(-8\cos y \sin x + 2\sin^2 x - 3\big) + 2y\sin y\big(\cos y + 8\sin x\big) +$$
$$+16\cos y \sin x + \sin^2 x - \sin^2 y + 4\,C\, + 1,$$

and P *is a polynomial in* x, y *and in sines and cosines of* x *and* y*:*

$$P = 16\big(x^6 y \sin x \sin y - x^5 y^2 \cos x \cos y +$$
$$+x^2 y^5 \sin x \sin y - xy^6 \cos x \cos y\big) + (\dots)$$

where by dots we denote lower order terms in x *and* y*. The functions* ψ_1 *and* ψ_2 *take the form*

$$\psi_1 = \frac{\omega_1}{Q}, \qquad \psi_2 = \frac{\omega_2}{Q}$$

and satisfy the equation

$$\widehat{H}\psi = \psi \qquad \text{with } \widehat{H} = -\Delta + \widehat{U}.$$

Let us assume that C *is negative and* $|C|$ *is sufficiently large, then* Q *has no zeroes and therefore the potential* $\widehat{U}$ *and the functions* ψ_1 *and* ψ_2 *are smooth. We have*

$$\widehat{U} = O\left(\frac{1}{r}\right), \quad \psi_1 = O\left(\frac{1}{r^2}\right), \quad \psi_2 = O\left(\frac{1}{r^3}\right), \quad \text{as } r = \sqrt{x^2+y^2} \to \infty. \quad (5)$$

and therefore ψ_1 *and* ψ_2 *lie in* $L_2(\mathbb{R}^2)$ *and are linearly independent eigenfunctions of the operator* $\widehat{H} = -\Delta + \widehat{U}$ *with the eigenvalue* $E = 1$*.*

Using various linear combinations of the solutions (4) one can easily construct multiparametric families of similar two-dimensional potentials $\widehat{U}$ with the asymptotics (5) and solutions ψ_i at the energy level $E = k^2$ and moreover to improve the decay of the eigenfunctions ψ_i. It is impossible to improve the decay of potentials due to already mentioned Kato's theorem [2].

We guess that by applying multiple iterations one may obtain such potentials with positive eigenvalues with higher multiplicity.

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